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On accelerated monotone iterations for numerical solutions of semilinear elliptic boundary value problems ☆

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Abstract

This paper is concerned with the computational algorithms for finite difference solutions of a class of semilinear elliptic boundary value problems. An accelerated monotone iterative scheme is presented by using the method of upper and lower solutions. The rate of convergence of the iterations is estimated by the infinity norm, and the rate of convergence is quadratic for a larger class of nonlinear functions, including monotone nonincreasing functions. An application is given to a logistic model problem in ecology.

Keywords: Monotone iteration; Quadratic convergence; Finite difference scheme; Elliptic boundary value problem; Method of upper and lower solutions

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1. Introduction

In the study of the numerical solutions of semilinear elliptic boundary value problems by finite difference methods, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations. Consider the following semilinear elliptic boundary value problem:

$$\begin{cases} -\nabla \cdot (D(x)\nabla u) + \mathbf{v} \cdot \nabla u = f(x,u), & x \in \Omega, \\ \alpha(x)\dfrac{\partial u}{\partial \nu} + \beta(x)u = g(x,u), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbf{R}^p$, $\mathbf{v} \cdot \nabla u \equiv v_1(x)\partial u/\partial x_1 + \cdots + v_p(x)\partial u/\partial x_p$, $\partial\Omega$ is the boundary of Ω and $\partial u/\partial \nu$ denotes the outward normal derivative of u on $\partial\Omega$. It is assumed that the function $D(x)$ is positive on $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, $\alpha(x)$ and $\beta(x)$ are nonnegative on $\partial\Omega$ with $\alpha(x) + \beta(x) > 0$. The functions $f(\cdot,u)$ and $g(\cdot,u)$, which in general are nonlinear in u, are assumed continuously differentiable in u. These functions and all the other prescribed functions are assumed continuous in their respective domains. By using the standard finite difference approximations for the differential and boundary operators in (1.1) and arranging the mesh points in the usual fashion, the resulting discrete problem of (1.1) becomes a system of nonlinear algebraic equations which in matrix form is given by

$$AU = F(U), \tag{1.2}$$

where if N denotes the total number of mesh points in $\overline{\Omega}$, then A is an $N \times N$ matrix, $U = (u_1, \ldots, u_N)^T$ is a solution vector, and $F(U)$ is a vector in the form

$$F(U) = (F_1(u_1), \ldots, F_N(u_N))^T, \quad F_i(u_i) = f(x_i, u(x_i)) + g(x_i', u(x_i')), \quad x_i \in \Omega, x_i' \in \partial\Omega. \tag{1.3}$$

The function $f(x_i, u(x_i))$ appears at the interior mesh points in Ω and $g(x_i', u(x_i'))$ appears at the boundary points and possibly neighboring mesh points of $\partial\Omega$. (See [1,2] for detailed derivations.)

For the system (1.2), a major concern is to obtain efficient computational algorithms for computing the solution. There are many iterative methods, and a well-known method is the method of upper and lower solutions with its associated monotone iterations. This method has been widely used for both continuous and discrete elliptic boundary value problems (cf. [2–12]). Most of the iteration processes used in the above works are of either Picard type or Jacobi–Gauss–Seidel type, and the rate of convergence of the iterations is only of linear order. In a recent article [13], an accelerated monotone iteration process for (1.2) is given. This method leads to quadratic convergence of the iterations, but the quadratic convergence requires that $F_i(u_i)$ is monotone nonincreasing in u_i. In this paper, we develop a different accelerated monotone iteration process for (1.2). The aim here is to relax the monotone condition on F_i so that the quadratic convergence of the iterations is ensured for a larger class of nonlinear functions F_i.

The outline of the paper is as follows. In Section 2, we present an accelerated monotone iterative scheme for the computation of solutions of (1.2). Section 3 is devoted to the quadratic convergence of the iterations. It is shown that the sequence converges quadratically to a solution of (1.2) for a larger class of nonlinear functions F_i, including the monotone nonincreasing case. The quadratic rate of convergence is estimated by the infinity norm. In Section 4, we give an application to a model problem in ecology.

2. Monotone iterative scheme

Motivated by the problem (1.1) we make the following hypothesis on A:

(H) the matrix $A \equiv (a_{i,j})$ is irreducible, and

$$a_{i,i} > 0, \qquad a_{i,j} \le 0 \ (i \ne j), \qquad \sum_{j=1}^{N} a_{i,j} \ge 0, \qquad i, j = 1, \ldots, N. \tag{2.1}$$

It is well-known that the relations in (2.1) can always be satisfied by the standard finite difference approximations for the differential and boundary operators in (1.1) if $\mathbf{v} \equiv \mathbf{0}$. In the case of $\mathbf{v} \not\equiv \mathbf{0}$ they are also satisfied by either taking an increment suitably small or using an upwind difference scheme for $\mathbf{v} \cdot \nabla u$ without any restriction on the increment (see [5,14]). The connectedness assumption of $\overline{\Omega}$ ensures that A is irreducible (see [15]). Therefore, the properties in (H) can always be satisfied. When the boundary condition in (1.1) is Dirichlet or Robin type (i.e. $\beta(x) \ne 0$ for some $x \in \partial\Omega$), the strict inequality in the last relation of (2.1) holds for at least one i, and in this situation A is a diagonally dominant M-matrix (see [15]). Moreover, the smallest eigenvalue of A, denoted by λ_0, is real and positive. Otherwise, if the equality in the last relation of (2.1) holds for all i, which corresponds to the pure Neumann boundary condition (i.e. $\beta(x) \equiv 0$), then $\lambda_0 = 0$ and A is singular. In our discussions we always include this special case. Under the Hypothesis (H) an important property of A is as follows.

Lemma 2.1. *Let Hypothesis (H) hold and let $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_N)$ be a diagonal matrix with $\min_i \gamma_i > -\lambda_0$. Then the inverse $(A + \Gamma)^{-1}$ exists and is nonnegative.*

Proof. Let $\delta > \max\{0, \max_i \gamma_i\}$, and I denote the identity matrix. Then by Hypothesis (H), the matrix $A+\delta I$ is a diagonally dominant M-matrix, and therefore the inverse $(A+\delta I)^{-1}$ exists and is nonnegative (cf. [15]). In addition, $0 \le \delta I - \Gamma \le (\delta - \min_i \gamma_i) I$ which implies

$$\rho((A + \delta I)^{-1}(\delta I - \Gamma)) \le (\delta - \min_i \gamma_i)\rho((A + \delta I)^{-1}) \qquad \text{(cf. [16])} \tag{2.2}$$

where $\rho(\cdot)$ denotes the spectral radius of the corresponding matrix. Since

$$\rho((A + \delta I)^{-1}) = \frac{1}{\lambda_0 + \delta} \qquad \text{(cf. [17])},$$

we have

$$\rho((A + \delta I)^{-1}(\delta I - \Gamma)) \le \frac{\delta - \min_i \gamma_i}{\lambda_0 + \delta} < 1.$$

Hence, the matrix

$$A + \Gamma = (A + \delta I)(I - (A + \delta I)^{-1}(\delta I - \Gamma))$$

is invertible and its inverse is nonnegative (see [16,17]). □

Definition 2.1. Two vectors $\widetilde{U}$ and $\widehat{U}$ are called a pair of ordered upper and lower solutions of (1.2) if $\widetilde{U} \ge \widehat{U}$ and

$$A\widetilde{U} \ge F(\widetilde{U}), \qquad A\widehat{U} \le F(\widehat{U}). \tag{2.3}$$

For a pair of ordered upper and lower solutions $\widetilde{U} = (\widetilde{u}_1, \ldots, \widetilde{u}_N)^T$ and $\widehat{U} = (\widehat{u}_1, \ldots, \widehat{u}_N)^T$ we define the sectors:

$$\langle \widehat{U}, \widetilde{U} \rangle = \{U \in \mathbf{R}^N : \widehat{U} \le U \le \widetilde{U}\}, \qquad \langle \widehat{u}_i, \widetilde{u}_i \rangle = \{u_i \in \mathbf{R} : \widehat{u}_i \le u \le \widetilde{u}_i\}, \qquad i = 1, 2, \ldots, N.$$

To compute the solution of (1.2) we use the following iterative scheme:

$$(A + \Gamma^{(m)})U^{(m+1)} = \Gamma^{(m)}U^{(m)} + F(U^{(m)}), \qquad m = 0, 1, 2, \ldots, \tag{2.4}$$

where $U^{(0)}$ is either $\widetilde{U}$ or $\widehat{U}$. The matrix $\Gamma^{(m)} = \mathrm{diag}(\gamma_1^{(m)}, \ldots, \gamma_N^{(m)})$ in (2.4) is defined by

$$\gamma_i^{(m)} = \begin{cases} c_i^{(m)}, & \text{if } c_i^{(m)} > -\lambda_0, \\ \delta, & \text{if } c_i^{(m)} \le -\lambda_0, \end{cases}$$

where δ is any positive constant and

$$c_i^{(m)} = \max\left\{ -\frac{\partial F_i}{\partial u_i}(u_i) : \underline{u}_i^{(m)} \le u_i \le \overline{u}_i^{(m)} \right\}, \qquad m = 0, 1, 2, \ldots.$$

The functions $\overline{u}_i^{(m)}, \underline{u}_i^{(m)}$ in the definition of $c_i^{(m)}$ are the respective components of $\overline{U}^{(m)}$ and $\underline{U}^{(m)}$ which are obtained from (2.4) with $U^{(0)} = \widetilde{U}$ and $U^{(0)} = \widehat{U}$, respectively.

To show that the iteration (2.4) is well-defined it is crucial that the sequences $\{\overline{U}^{(m)}\}$, $\{\underline{U}^{(m)}\}$ possess the property $\overline{U}^{(m)} \ge \underline{U}^{(m)}$ for every m. This is given by the following lemma.

Lemma 2.2. *Let $\widetilde{U}$ and $\widehat{U}$ be a pair of ordered upper and lower solutions of (1.2), and let Hypothesis (H) hold. Then the sequences $\{\overline{U}^{(m)}\}$, $\{\underline{U}^{(m)}\}$ given by (2.4) with $\overline{U}^{(0)} = \widetilde{U}$, $\underline{U}^{(0)} = \widehat{U}$ are well-defined and possess the monotone property*

$$\widehat{U} \le \underline{U}^{(m)} \le \underline{U}^{(m+1)} \le \overline{U}^{(m+1)} \le \overline{U}^{(m)} \le \widetilde{U}, \qquad m = 0, 1, 2, \ldots. \tag{2.5}$$

Moreover, for each $m = 1, 2, \ldots$, $\overline{U}^{(m)}$ and $\underline{U}^{(m)}$ are a pair of ordered upper and lower solutions of (1.2).

Proof. By the definition of $\Gamma^{(m)}$ and Lemma 2.1, the inverse $(A + \Gamma^{(m)})^{-1}$ exists and is nonnegative as soon as $\Gamma^{(m)}$ is known. The remainder of the proof is similar to that in [13]. □

Theorem 2.1. *Let the conditions in Lemma 2.2 hold. Then the sequence $\{\overline{U}^{(m)}\}$ converges monotonically from above to a maximal solution $\overline{U}$ of (1.2) in $\langle \widehat{U}, \widetilde{U} \rangle$, and the sequence $\{\underline{U}^{(m)}\}$ converges monotonically from below to a minimal solution $\underline{U}$ of (1.2) in the same sector. Moreover,*

$$\widehat{U} \le \underline{U}^{(m)} \le \underline{U}^{(m+1)} \le \underline{U} \le \overline{U} \le \overline{U}^{(m+1)} \le \overline{U}^{(m)} \le \widetilde{U}, \qquad m = 0, 1, 2, \ldots. \tag{2.6}$$

If, in addition,

$$\frac{\partial F_i}{\partial u_i}(u_i) < \lambda_0, \qquad u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle, \qquad i = 1, 2, \ldots, N, \tag{2.7}$$

then $\overline{U} = \underline{U}$ and is the unique solution of (1.2) in $\langle \widehat{U}, \widetilde{U} \rangle$.

Proof. The first part of the theorem follows from the same argument as that in [13]. To prove the second part we observe that $\overline{U}$ and $\underline{U}$ satisfy that

$$(A - C)(\overline{U} - \underline{U}) = 0, \qquad C = \mathrm{diag}\left(\frac{\partial F_1}{\partial u_1}(\xi_1), \ldots, \frac{\partial F_N}{\partial u_N}(\xi_N) \right),$$

where $\xi_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle$ $(i = 1, 2, \ldots, N)$. By (2.7) and Lemma 2.1, the inverse $(A - C)^{-1}$ exists, and therefore $\overline{U} = \underline{U}$. □

3. Quadratic convergence of the sequences

In this section we show the quadratic convergence of the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$. Assume that for each $i = 1, 2, \ldots, N$, $F_i(u_i)$ is a C^2-function of $u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle$. Define

$$M_1 = \max_i \max \left\{ \left| \frac{\partial^2 F_i}{\partial u_i^2}(u_i) \right| : u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle \right\}, \qquad M_2 = \min_i \min \left\{ -\frac{\partial F_i}{\partial u_i}(u_i) : u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle \right\}. \quad (3.1)$$

Under the condition (2.7) we have from Lemma 2.1 that the inverse $(A + M_2 I)^{-1}$ exists and is nonnegative. Moreover, we have from Theorem 2.1 that $\overline{U} = \underline{U} (\equiv U^*)$ and $U^* = (u_1^*, \ldots, u_N^*)^T$ is the unique solution of (1.2) in $\langle \widehat{U}, \widetilde{U} \rangle$.

Theorem 3.1. *Let the conditions in Lemma 2.2 and the condition (2.7) be satisfied, and let U^* be the unique solution of (1.2) in $\langle \widehat{U}, \widetilde{U} \rangle$. Then*

$$\|\overline{U}^{(m+1)} - U^*\|_\infty + \|\underline{U}^{(m+1)} - U^*\|_\infty \le \sigma (\|\overline{U}^{(m)} - U^*\|_\infty + \|\underline{U}^{(m)} - U^*\|_\infty)^2, \quad m = 0, 1, \ldots, \quad (3.2)$$

where $\sigma = M_1 \|(A + M_2 I)^{-1}\|_\infty$.

Proof. The result (3.2) is different from that in [13]. We give a proof for the present result. Consider the sequence $\{\overline{U}^{(m)}\}$. By (1.2) and (2.4),

$$(A + \Gamma^{(m)})(\overline{U}^{(m+1)} - U^*) = \Gamma^{(m)}(\overline{U}^{(m)} - U^*) + F(\overline{U}^{(m)}) - F(U^*), \qquad m = 0, 1, 2, \ldots. \quad (3.3)$$

Since the condition (2.7) holds we have from the definition of $\Gamma^{(m)}$ and the mean-value theorem that

$$\Gamma^{(m)} = \operatorname{diag}\left(-\frac{\partial F_1}{\partial u_1}(\xi_1^{(m)}), \ldots, -\frac{\partial F_N}{\partial u_N}(\xi_N^{(m)}) \right) \quad (3.4)$$

where $\xi_i^{(m)} \in \langle \underline{u}_i^{(m)}, \overline{u}_i^{(m)} \rangle$. Again by the mean-value theorem, there exists $\eta_i^{(m)} \in \langle u_i^*, \overline{u}_i^{(m)} \rangle$ such that

$$F(\overline{U}^{(m)}) - F(U^*) = C^{(m)}(\overline{U}^{(m)} - U^*), \qquad C^{(m)} = \operatorname{diag}\left(\frac{\partial F_1}{\partial u_1}(\eta_1^{(m)}), \ldots, \frac{\partial F_N}{\partial u_N}(\eta_N^{(m)}) \right), \quad (3.5)$$

and there exists $\theta_i^{(m)}$ between $\xi_i^{(m)}$ and $\eta_i^{(m)}$ such that

$$\frac{\partial F_i}{\partial u_i}(\eta_i^{(m)}) - \frac{\partial F_i}{\partial u_i}(\xi_i^{(m)}) = \frac{\partial^2 F_i}{\partial u_i^2}(\theta_i^{(m)})(\eta_i^{(m)} - \xi_i^{(m)}).$$

Since $|\eta_i^{(m)} - \xi_i^{(m)}| \le \overline{u}_i^{(m)} - \underline{u}_i^{(m)}$, the above conclusions imply that

$$0 \le (\overline{U}^{(m+1)} - U^*) \le M_1 \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty (A + \Gamma^{(m)})^{-1} (\overline{U}^{(m)} - U^*). \quad (3.6)$$

Since $M_2 > -\lambda_0$ and $\Gamma^{(m)} \ge M_2 I$, we have from Lemma 2.1 that $0 \le (A + \Gamma^{(m)})^{-1} \le (A + M_2 I)^{-1}$ (cf. [16]). This result and (3.6) lead to

$$0 \le (\overline{U}^{(m+1)} - U^*) \le M_1 \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty (A + M_2 I)^{-1} (\overline{U}^{(m)} - U^*) \quad (3.7)$$

which implies that

$$\|\overline{U}^{(m+1)} - U^*\|_\infty \le \sigma \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty \|\overline{U}^{(m)} - U^*\|_\infty. \quad (3.8)$$

A similar argument leads to

$$\|\underline{U}^{(m+1)} - U^*\|_\infty \le \sigma \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty \|\underline{U}^{(m)} - U^*\|_\infty. \tag{3.9}$$

An addition of (3.8) and (3.9) yields (3.2). □

Theorem 3.2. *Let the conditions in Theorem 3.1 hold. Then*

$$\|\overline{U}^{(m+1)} - U^*\|_\infty \le \sigma \|\overline{U}^{(m)} - U^*\|_\infty^2, \qquad m = 0, 1, 2, \ldots, \tag{3.10}$$

if $\partial^2 F_i/\partial u_i^2 \le 0$ *for* $u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle$, *and*

$$\|\underline{U}^{(m+1)} - U^*\|_\infty \le \sigma \|\underline{U}^{(m)} - U^*\|_\infty^2, \qquad m = 0, 1, 2, \ldots, \tag{3.11}$$

if $\partial^2 F_i/\partial u_i^2 \ge 0$ *for* $u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle$, *where* σ *is the same as that in* (3.2).

Proof. Consider the case $\partial^2 F_i/\partial u_i^2 \le 0$ for $u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle$. In this case, $\xi_i^{(m)} = \overline{u}_i^{(m)}$ where $\xi_i^{(m)}$ is the intermediate value that appeared in (3.4). Since $\eta_i^{(m)} \in \langle u_i^*, \overline{u}_i^{(m)} \rangle$ where $\eta_i^{(m)}$ is the intermediate value that appeared in (3.5), we see that $|\eta_i^{(m)} - \xi_i^{(m)}| \le \overline{u}_i^{(m)} - u_i^*$. The argument in the proof of (3.8) shows that (3.10) holds. The proof of (3.11) is similar. □

Remark 3.1. Theorem 3.1 gives a quadratic convergence for the sum of the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ in the infinity norm, while Theorem 3.2 shows that if F_i is a concavity or convexity function, then one of the two sequences converges quadratically in the infinity norm to the unique solution U^*. The advantages of the above results over those in [13] are twofold: (i) the quadratic convergence of the sequences is ensured for the larger class of nonlinear functions F_i, including the case where $F_i(u_i)$ is monotone nonincreasing in u_i, and (ii) the rate of convergence is explicitly estimated by the infinity norm.

4. Application

Let Ω be a rectangular domain in $\mathbf{R}^2$. We consider a Logistic model in ecology (cf. [12]):

$$\begin{cases} -\Delta u = \gamma u(1-u) + q(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where γ is a positive constant and $q(x)$ is a possible internal source. For physical reasons we assume that $q(x) \ge 0$ and $g(x) \ge 0$. Let h be the increment. By using the standard finite difference approximation for the differential operator the resulting discrete problem of (4.1) is given in compact form (1.2), where $A = h^{-2}\mathrm{tridiag}(C_i, A_i, C_i)$ is a block tridiagonal matrix with

$$A_i = \mathrm{tridiag}(-1, 2, -1), \qquad C_i = \mathrm{diag}(-1, \ldots, -1),$$

and the components of $F(U)$ have the form $F_i(u_i) = \gamma u_i(1-u_i) + q(x_i) + g^*(x_i)$. The function $g^*(x_i)$ is associated with the boundary function $g(x_i)$ and appears only on the neighboring mesh points of $\partial\Omega$. It is clear that the matrix A satisfies all the conditions in Hypothesis (H) and the smallest eigenvalue λ_0 of A is positive. (In the one-dimensional case $\Omega = (0,1)$, $\lambda_0 = (4/h^2)\sin^2(\pi h/2) > 4$). Since $q(x) \ge 0$ and $g(x) \ge 0$, $\widehat{U} \equiv 0$ is a lower solution. On the other hand, for any constant K satisfying $\gamma K(K-1) \ge M_q + M_g$, the constant vector $\widetilde{U} = (K, \ldots, K)^T$ is an upper solution, where $M_q = \max_{x\in\overline{\Omega}} q(x)$ and $M_g = \max_{x\in\partial\Omega} g(x)$. Since $F_i(u_i)$ is not monotone for all $u_i \in \langle 0, K \rangle$, the quadratic convergence of the iterations given in [13] is not ensured for each γ. But nevertheless we do have a monotone iterative process of the form (2.4) that leads to two sequences which converge quadratically to the unique solution U^* of the problem in $\langle \widehat{U}, \widetilde{U} \rangle$ provided $\gamma < \lambda_0$, because all conditions of Theorems 3.1 and 3.2 are satisfied.

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